Generation of Large Number-Path Entanglement Using Linear Optics and Feed-Forward

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We show how an idealised measurement procedure can condense photons from two modes into one, and how, by feeding forward the results of the measurement, it is possible to generate efficiently superpositions of components for which only one mode is populated, commonly called "N00N states". For the basic procedure, sources of number states leak onto a beam splitter, and the output ports are monitored by photodetectors. We find that detecting a fixed fraction of the input at one output port suffices to direct the remainder to the same port with high probability, however large the initial state. When instead photons are detected at both ports, Schrödinger cat states are produced. We describe a circuit for making the components of such a state orthogonal, and another for subsequent conversion to a N00N state. Our approach scales exponentially better than existing proposals. Important applications include quantum imaging and metrology.

The fundamental limits to optical detection for metrology and imaging are quantum mechanical [1]. Of particular interest for reaching such quantum limits are path-entangled states of photons of the form $|N0\rangle + e^{i\phi}|0N\rangle$, in a basis of photon-number states, commonly referred to as "N00N" states. A variety of applications have been suggested [2]. For lithography [3] and microscopy [4], N00N state light would be used together with multiphoton absorbers to achieve enhanced resolution. This is because the de Broglie wavelength for an N-photon state is a factor $1/N$ smaller than the wavelength associated with the single photon, and the absorption rate scales linearly with the incident intensity, rather than as the Nth power. Regarding applications to precision metrology, whereby an interferometric setup is used to measure small phase shifts, N00N states achieve the Heisenberg limit, for which the phase uncertainty scales as $1/N$ [5, 6, 7], and entanglement is a fundamental requirement for achieving this limit. It has been rigorously demonstrated that the cost of improving sensitivity (without using entanglement) is higher intensities or longer coherence times [8]. Classically the shot-noise limit applies, attained for example by laser light, for which the uncertainty scales as $1/\sqrt{N}$, already a restraint in applications such as magnetometry [9] and gyroscopy [10].

However, building a source of N00N states beyond two photons is challenging. Three, four and six photon experiments have been reported [11, 12, 13], but only in the first two references were N00N states generated. In theory a source could be made using a nonlinear crystal [14]. However, the required optical nonlinearity is not readily available. An alternative is a non-deterministic approach using linear optics, wherein the desired state is generated on condition of a specific outcome at photodetectors. A variety of schemes have been suggested which typically rely on conditional destructive interference [15, 16, 17].

However, so far none of these scales efficiently, that is they all share the feature that exponentially decreasing success probabilities outweigh the possible gains. Noting that quantum algorithms, exhibiting polynomial and exponential speedups over their classical counterparts, may be implemented scalably in a linear-optics approach [18], we expect that it should be possible to do better. In this Article we address this challenge by adapting a conceptually simple measurement procedure. Our method is as follows. First, we aim to minimise the negative effects of back-action in a sequence of detections, whereby earlier measurements affect the outcomes of later ones. Next, we engineer output states that closely approximate the ideal case. Finally, we exploit feed-forward, for which circuits are actively switched in response to previous photodetections. Feed-forward is an essential ingredient of linear-optics based quantum computing, but is not used in previous proposals for engineering N00N states.

We begin by considering the thought experiment depicted in Fig. 1(a). Here, cavity modes labeled A and B are assumed to start with a well-defined photon number N. They are each coupled to an external mode by a weakly transmissive mirror, and these modes are combined at a 50:50 beam splitter, and then subject to partial photodetection. The beam splitter acts to make the origin of the photons indistinguishable. When a photon is registered at the left or right photodetector labeled D_L or D_R, the transformation is given by the Kraus operators $\hat{L} = (\hat{a} - \hat{b})/\sqrt{2}$ or $\hat{R} = (\hat{a} + \hat{b})/\sqrt{2}$ respectively (where $\hat{a}$ and $\hat{b}$ are the annihilation operators for modes A and B). To obtain the corresponding probabilities it is necessary to normalise by the total photon number prior to detection. We suppose now that a string of detections occur only at D_r, say (by adjusting the path length difference of the cavities between detections, with a phase shifter, the same state for the cavity modes can be obtained in every case). After this, the detectors are removed and the system evolves to a final state with all the remaining population at the output ports. Denoting by $|\psi_{AB}\rangle$ the state of modes A

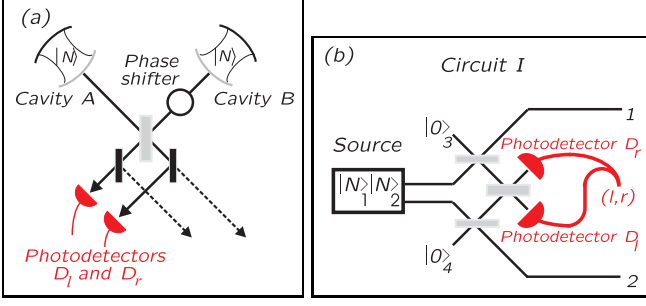*Electronic address: hcable@lsu.edu

FIG. 1: Two measurement-based procedures, inducing relative phase correlations between the principal modes, each one with N photons at the start. In (a), population leaks from cavity modes A and B into external modes, which are combined at a beam splitter. Photons are detected one at a time by photodetectors D_l and D_r. In (b), all modes are propagating. Beam splitters couple some fraction f from the principal modes one and two into ancillae modes three and four, which are initially the vacuum. The ancillae are subsequently combined at a beam splitter, and subjected to number-resolving photodetection at D_l and D_r.

and B after r initial detections, we find that $|\psi_{AB}\rangle = \hat{R}^r |N\rangle |N\rangle / \sqrt{\langle N| \langle N| \left(\hat{R}^\dagger\right)^r \left(\hat{R}\right)^r |N\rangle |N\rangle}$, normalising to unity. The probability $P_{\rm cond}$ of finding all the remaining photons "condensed" at the right output port (and none at the left output port) is as follows,

$$
\begin{aligned}
P_{\rm cond} &= \langle \psi_{AB}| \left(\hat{R}^\dagger\right)^S \left(\hat{R}\right)^S |\psi_{AB}\rangle / S! \\
&= \left(^{2N}C_N\right)^2 / \left[2^S \sum_{k=0}^{r} \left(^rC_k\right)^2 \left(^SC_{N-k}\right)\right],
\end{aligned}
$$

where $S \equiv 2N - r$ denotes the total remaining photon number, C denotes a binomial coefficient, and we assume that $r < N$. Evaluating the value of $P_{\rm cond}$ numerically for initial states of increasing size, we find that its value is determined asymptotically by the proportion of the input that is measured. For example, setting r either as one quarter or one third of $2N$ suffices for $P_{\rm cond} > 0.6$ or $P_{\rm cond} > 0.7$, respectively.

We have found for our thought experiment, that later detections tend to strongly reinforce earlier ones. Hence, the effect of measurement back-action here is useful for state engineering, and in what follows we adapt the measurement process for $N00N$ state generation. We divide our analysis into three stages. First, we translate our thought experiment into a mathematically equivalent procedure based purely on linear optics, and consider the general case for which photons are detected at both photodetectors. The localisation phenomena resulting from this measurement process, have been studied extensively in the context of the debate over the existence of absolute optical coherence in common quantum-optical experiments [19]. It has been demonstrated that well-defined correlations in the relative optical phase evolve, for the remaining population, and play a central role in the ongo-

ing dynamics. Hence, in the second stage of our analysis, we investigate simple procedures for manipulating phase correlations, and relate states with well-defined correlations in the relative phase and $N00N$ states. Finally, we identify a method based on feed-forward to enable $N00N$ states to be generated efficiently.

First, we translate our thought experiment into a mathematically-equivalent procedure, based on linear optics, as depicted in Fig. 1(b). We label this optical circuit, Circuit I. Here all modes are propagating, and a source is assumed to supply dual Fock states $|N\rangle|N\rangle$ to the principal modes one and two. Beam splitters of reflectance f couple modes one and two to ancillae modes three and four, which are combined at a 50:50 beam splitter. They are then measured by number-resolving photodetectors labeled D_l and D_r, where on average a fraction f of the input photons are registered. We now consider the state $|\psi_{l,r}\rangle$ generated in modes one and two after l photons are registered at D_l and r at D_r. Following Ref. [20], it is convenient to adopt a representation in terms of coherent states, which are of the form, $|\alpha\rangle \equiv ||\alpha| \exp(i\theta)\rangle \propto \sum_{k=0}^{\infty} \sqrt{|\alpha|^{2k}/k!} \exp(ik\theta) |k\rangle$ in a basis of Fock states. It has been shown that,

$$
\begin{aligned}
|\psi_{l,r}\rangle \propto \int_0^{2\pi} \int_{-\pi}^{\pi} d\theta_{\rm av} d\Delta \exp\left(-iS\theta_{\rm av}\right) \times \\
\left[G(\Delta - \Delta_0) + \exp\left(i\sigma\right) G(\Delta + \Delta_0)\right] |\alpha_1\rangle|\alpha_2\rangle, (1)
\end{aligned}
$$

where $S \equiv 2N - l - r$, $\alpha_j = |\alpha_j| \exp(i\theta_j)$, $\theta_{\rm av} = (\theta_1 + \theta_2)/2$ and $\Delta \equiv \theta_2 - \theta_1$. The superposition phase σ takes the value $l\pi$, and hence the measurement record must be known exactly. The scalar function $G(X)$ is given to good approximation by the Gaussian expression $\exp\left[-(l+r)X^2/4\right]$. The total photon number is equal to S. There are well-defined correlations in the relative-optical phase parameter Δ at values plus and minus Δ_0, determined only by the ratio of l to r. These correlations are multi-valued whenever photons are registered at both photodetectors, and a Schrödinger cat state is generated. We can see that cats are generated as a result of the symmetry of the setup. Specifically, $\hat{L}$ and $\hat{R}$ are invariant under an exchange of the labeling of the modes, a transformation which reverses the sign of the relative phase. The generation of cat states therefore also requires precise phase stability between the modes. Turning to the source, we see that the state of the input can be a mixture of the form $\sum_N P_N |N\rangle|N\rangle\langle N|\langle N|$, since Δ_0 is independent of N, and standard linear optical elements obey a superselection rule for the photon number. Several two-mode squeezing processes strongly suppress relative number fluctuations, and hence might serve as practical sources of light described by these mixed states.

For the second stage in our analysis, we identify the outputs of Circuit I as examples of quantum reference frames — reference frames for a classically defined parameter composed of finite quantum resources. Quantum reference frames are subject to depletion and degradation as they are used, and are currently of interest for proto-

cols in the field of quantum information, in which they are regarded as a resource [21]. By making an analogy to classical phase references we can now identify simple ways in which states of the form Eq. (1) can be manipulated. For the current purposes we can assume that a large number of detections have been performed and define,

$$|\psi_\infty(\Delta_0)\rangle \propto \int_0^{2\pi} d\theta \exp(-iS\theta)|\alpha\rangle|\alpha\exp(i\Delta_0)\rangle, \quad (2)$$

where $\alpha = |\alpha|e^{i\theta}$, for a state with a total photon number S and a relative phase of Δ_0 (assumed to be normalized). Relative phase correlations between more than two modes are transitive and are transformed additively by phase shifters. A phase reference can be extended to additional modes by combining it with the vacuum at a beam splitter. As an example, a 50:50 beam splitter, which we denote here by U_{bs}, beating light in a Fock state with S photons against the vacuum yields, $U_{\mathrm{bs}}|S\rangle|0\rangle \propto |\psi_\infty(0)\rangle$, and $U_{\mathrm{bs}}|0\rangle|S\rangle \propto |\psi_\infty(\pi)\rangle$. Therefore, we see that a simple circuit, consisting only of a beam splitter and a phase shifter, can convert a cat state generated by Circuit I to a $N00N$ state, whenever the relative phase correlations differ by π. This happens when $l = r$ and $\Delta_0 = \pi/2$. We label this circuit, Circuit III (anticipating an intermediate process modifying the cat states for the general case).

Before proceeding to the final stage of our analysis, we consider a simple $N00N$-state generator, that attempts to convert every cat state generated by Circuit I using Circuit III. This method might be expected to yield close approximations to $N00N$ states, whenever the the relative phases of the cat state are close to plus and minus $\pi/2$. The situation is summarized in Fig. 2(a). To measure the quality of the output state we adopt the fidelity, denoting it by F. For the measurement-induced condensation, considered at the start, F takes the same same value as P_{cond}. For schemes generating $N00N$ states, it is necessary to account for the phase of the superposition, and we define $F = \max_\phi \left|\left(\langle 0S| + \exp(-i\phi)\langle S0|\right)|\psi_{\mathrm{output}}\rangle\right|^2/2$, where S is the total photon number of the state $|\psi_{\mathrm{output}}\rangle$. Evaluating F for our $N00N$-state generator, when Circuit I generates a cat state with relative phase components at $\pm\Delta_0$ and total photon number S, we find to first approximation that $F \sim \cos^{2S}[(\Delta_0 - \pi/2)/2]$. As with other proposals, this scheme in fact scales *exponentially poorly* whenever the relative phase correlations are less than π apart, as is typically the case. Inspecting the overlap for different relative phase components, as in Eq. (2) with total photon number S, we find that $\left|\langle\psi_\infty(\Delta_1)|\psi_\infty(\Delta_2)\rangle\right| = \left|\cos[(\Delta_2 - \Delta_1)/2]\right|^S$. The poor scaling can be attributed to the non-orthogonality of the cat state components.

We now proceed to the final stage of our analysis. Our previous $N00N$-state generator is effective when Circuit I
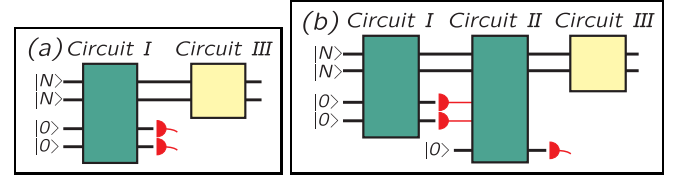

FIG. 2: (a) and (b) illustrate complete $N00N$ state generators in outline. Circuit I produces Schrödinger cat states with two relative phase components non-deterministically (green). The generators are terminated by a fixed unitary (yellow) circuit consisting of beam splitters and phase shifters. In (b) the cat states are corrected, using an additional measurement process conditioned on the previous detection outcomes.

generates cat states with components which are orthogonal. However, this occurs with low probability. It is not possible to improve the situation with any combination of (idealised) beam splitters and phase shifters, since these implement unitary transformations. Hence, we now devise a circuit, labeled Circuit II, to make input cat components orthogonal, using additional processes of measurement and feed-forward. This more sophisticated scenario is depicted in Fig. 2(b). To identify a suitable circuit, we investigate how a beam splitter transforms phase references, starting with two classical fields. Here the field in each mode is represented by a complex number, with the square amplitude corresponding to the intensity, and the phase to the optical one. A 50:50 beam splitter, configured so as not to impart additional phase shifts to the modes, outputs two classical fields described by the sum and difference of the values for the inputs (altering both the square amplitudes and the phases). If the input has a relative phase of 0 or π, and equal intensities for each mode, the population is transferred entirely into one mode. On the other hand, if the input has a relative phase of plus or minus $\pi/2$, and equal intensities in each mode, the relative phase and intensities are preserved. Moving to the quantum case, we consider the action of the beam splitter for a state, defined as in Eq. (2), with a relative phase of Δ_0, a total photon number S, and an intensity $S/2$ in each mode. Computing the final state explicitly, we find a scenario similar to the classical case,

$$U_{\mathrm{bs}}|\psi_\infty\rangle \propto \int_0^{2\pi} d\theta \exp(-iS\theta) \times$$
$$|\sqrt{I_1}\exp(i\theta)\rangle|\sqrt{I_2}\exp[i(\theta \pm \pi/2)]\rangle. \quad (3)$$

This has an intensity $SI_1/(I_1 + I_2) = [1 - \cos(\Delta_0)]/2$ in mode one and $SI_2/(I_1 + I_2) = S[1 + \cos(\Delta_0)]/2$ in mode two, and a relative phase of plus $\pi/2$ when $0 < \Delta_0 \leq \pi/2$ and of minus $\pi/2$ when $-\pi/2 \leq \Delta_0 < 0$ (we consider cases for which the intensity is increased in favour of mode two).

The symmetry of the beam splitter transformation makes it useful for altering the cats generated by Circuit I, so that the relative phases are different by π. However, it creates a difference in the intensities between the modes. To correct this, we propose beating mode two

against the vacuum, so as to to move the difference of the intensities to an ancillary mode, which can be removed by a photodetection. This method depends critically on feeding forward the result of the detections performed by Circuit I, so that a variable beam splitter can be set according to the value of Δ_0. A variable beam splitter can be implemented with 50:50 beam splitters and variable phase shifters. The cost of correction is a decrease in the total photon number, which varies non-deterministically. As can be seen from Eq. (3), a fraction of $\cos(\Delta_0)$ of the photons are lost on average. Overall, Circuits I through III constitute a complete $N00N$-state generator. Additional mathematical analysis is given in the supplementary online text. Runs for which Circuit I fails to generate a Schrödinger cat state, or too many photons are lost in the detection process are discarded. The fidelities at the output are, on average, 0.87, 0.94 or 0.98, when a fraction of one third, one half, or two third respectively of the input photons are detected by Circuit I. Higher fidelities are possible when the photon number at the input is small. If allowance is made for sufficient input photons to be detected by Circuit I, and a further half to be detected in Circuit II, the probability of failure is not too large.

Finally, we suggest some possibilities for experimental implementation. For the source, we propose an optical parametric oscillator setup for which the two mode squeezed output of an optical parametric amplifier is enhanced by a cavity [22]. Note, however, that the current purposes require twin beams of a much lower intensity than is typical in many experiments, and that the beams must be rendered frequency degenerate. Techniques of feed-forward and photodetection are being developed with a view to quantum information technologies [23, 24]. For the source, an important problem is imperfect correlation between the modes. If, for example, two independent lasers of equal intensity provide the input, the scheme generates the intended relative phase correlations, but no entanglement [25]. Photodetectors are subject to loss and dark counts. Losses will act to degrade the source, reducing the relative number correlation and increasing the uncertainty in the total photon number. Dark counts are more problematic, mixing over the phase for the superposition in Eq. (1). An alternative suggestion is using trapped bosonic atoms. One possibility might be to work in a regime for which the atomic wave-packets are much longer than the typical scattering length, as proposed in Ref. [26]. Another is to use Bose-Einstein condensates, for which a variety of coherent operations have been demonstrated. Number-resolved condensates might be obtained from the Mott Insulator phase, while relative-number squeezing can be achieved by different techniques.

In conclusion, we have proposed for the first time a linear-optics based scheme that generates large $N00N$ states efficiently, the photon number at the output scaling with that of the source — all the while maintaining high fidelities, high success probabilities and a fixed num-ber of circuit components. As well as being of immediate interest for a range of applications, our results have connections with other topics. For example, the scaling we derive for our measurement-induced condensation procedure is of relevance to the study of the interference of light from independent sources and localizing relative optical phase, phenomena with analogs in different physical systems [27]. We have left as an open question the extent to which the scaling can be attributed to Bose statistics. Regarding our $N00N$-state generators, the creation of macroscopic entangled states is of interest for exploring the quantum-classical transition. Finally, our study of Schrödinger cat states may have application to quantum computing, where Schrödinger cat states, defined for one mode only, have been proposed to encode qubits, which may be manipulated using standard experimental techniques [28].

Acknowledgements

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Supplementary Material: Methods

In these supplementary notes, we provide further analysis of our $N00N$-state generator, consisting of Circuits I, II and III, as depicted in outline in Fig. 2(b). First, we specify notation for beam splitters, phase shifters and states with well-defined relative phase correlations. For the lossless beam splitter, we choose a notation which makes explicit the "rotation" performed by such a device. A beam splitter with transmittance τ and reflectance $(1 - \tau)$ acts to transform the annihilation operators $\hat{o}_j$ for modes labeled j, according to the relations,

$$\begin{pmatrix} \hat{o}_1 \\ \hat{o}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \begin{pmatrix} \hat{o}_1 \\ \hat{o}_2 \end{pmatrix}$$

with angular parameter γ, where $\tau = \cos^2(\gamma)$ and $0 \leq \gamma \leq \pi/2$. We denote this transformation $U_{\text{bs}}(\gamma)$, and we include, where necessary, phase shifts of χ at the input port and $-\chi$ at the output port of the first mode, so that $U_{\text{bs}}(\gamma, \chi) \equiv \exp\left(\gamma \exp(i\chi)\hat{o}_1\hat{o}_2^\dagger - \gamma \exp(-i\chi)\hat{o}_1^\dagger\hat{o}_2\right)$. For example, $U_{\text{bs}}(\arccos(\sqrt{\tau}), \pi/2)$ corresponds to a symmetric beam splitter. We denote a phase shift transformation on mode j, $\exp\left(i\hat{o}_j{}^\dagger\hat{o}_j\chi\right)$, by $U_{\text{ps}}(\chi)$. For a state defined, as in Eq. (2), with a total photon number S and relative phase Δ_0, it is helpful to incorporate a phase factor $\exp\left(-iS\Delta_0/2\right)$ into the normalisation (making the definition symmetric between the modes). We then adopt the following notation for a normalised Schrödinger cat state,

$$|\psi_{\text{cat}}(\Delta_0, \Lambda)\rangle \propto |\psi_\infty(\Delta_0)\rangle + \exp(i\Lambda)|\psi_\infty(-\Delta_0)\rangle,$$

having components with relative phases plus and minus Δ_0, a phase for the superposition Λ (with the overall normalisation constant assumed positive).

Next, we elaborate on the sequence of operations performed by our $N00N$-state generator. We assume the final state should have at least P photons, and that the correlations in the relative phase are ideal. For the first step, Circuit I, depicted in Fig. 1(b), implements the transformation,

$$|l, r\rangle\langle l, r|_{3,4} U_{\text{bs}}\left(\frac{\pi}{4}\right)_{3,4} U_{\text{bs}}\left[\text{asin}\left(\sqrt{f}\right)\right]_{1,3} U_{\text{bs}}\left[\text{asin}\left(\sqrt{f}\right)\right]_{2,4},$$

conditioned on the detection of l photons in mode 3 and r photons in mode 4. A dual Fock state from the source evolves to a cat state according to,

$$\text{Circuit I}: |N\rangle_1|N\rangle_2|0\rangle_3|0\rangle_4 \longrightarrow |\psi_{\text{cat}}(\Delta_0, l\pi)\rangle_{1,2}.$$

This cat state has relative phase components with values plus and minus $\Delta_0 \equiv 2\arccos\left[\sqrt{r/(l+r)}\right]$, and total photon number $2N - l - r$. Runs for which l or r are zero must be discarded. It has been shown that values for the relative phase are generated with approximately equal frequency across the range [25], and hence these failure events do not affect the scaling of the generator.

Next, Circuit II acts to transform the relative phase correlations, to plus and minus $\pi/2$, in every case. When $r \geq l$, the relative phase correlations lie in the range $[-\pi/2, \pi/2]$, and a 50:50 beam splitter acting on the principal modes corrects the relative phase correlations, while increasing the intensity in mode two (and decreasing it in mode one). To achieve the same outcome when $l < r$, we suppose that a phase shift of π is applied in advance (on either mode). This transforms the cat state generated by Circuit I as,

$$U_{\text{ps}}(\pi)|\psi_{\text{cat}}(\Delta_0(l, r), l\pi)\rangle \propto |\psi_{\text{cat}}(\Delta_0(r, l), r\pi)\rangle.$$

Next, a beam splitter, with transmittance $[1 - \cos(\Delta_0)]/[1 + \cos(\Delta_0)]$, transfers the difference of the intensities to the ancillary mode five. A circuit for implementing the variable beam splitter is given by the relation,

$$U_{\text{bs}}(\gamma)_{2,5} \equiv$$
$$U_{\text{bs}}(\pi/4, \pi/2)_{2,5} U_{\text{ps}}(\gamma)_5 U_{\text{ps}}(-\gamma)_2 U_{\text{bs}}(\pi/4, -\pi/2)_{2,5}.$$

A photodetector measures Q photons in mode 5. Overall, Circuit II implements the transformation,

$$|Q\rangle\langle Q|_5 U_{\text{bs}}\{\arccos[\tan(\Delta_0/2)]\}_{2,5} U_{\text{bs}}(\pi/4)_{1,2}.$$

The cat state evolves as,

$$\text{Circuit II}:$$
$$|\psi_{\text{cat}}(\Delta_0, l\pi)\rangle \longrightarrow |\psi_{\text{cat}}[\pi/2, l\pi + (2N - l - r - Q)\pi/2]\rangle.$$

We derived a full probability distribution for the outcomes of the photodetection performed by Circuit II,

$$\text{Prob}(Q = 0, \cdots, S - 1) = \frac{1}{1 + (-1)^l \cos^S(\Delta_0)} \, {}^S C_Q \, [1 - \cos(\Delta_0)]^{S-Q} \cos^Q(\Delta_0)$$

$$\text{Prob}(Q = S) = \frac{1}{1 + (-1)^l \cos^S(\Delta_0)} \frac{\left[1 + (-1)^l\right]^2}{2} \cos^S(\Delta_0) \ ,$$

where $S = 2N - l - r$ is the total photon number prior to detection, and C denotes a binomial coefficient. This probability distribution is approximately binomial, and the expected number of detections is $S \cos(\Delta_0)$. If too many photons are lost in Circuits I and II the run must be aborted. The probability of this can be made small by taking $P \simeq N(1 - f)$. In principle, excess photons can be removed by an additional process, similar to Circuit I.

Finally, Circuit III implements the unitary transformation,

$$U_{\mathrm{bs}}(\pi/4, \pi)_{1,2} U_{\mathrm{ps}}(\pi/2)_2 .$$

The corrected cat state evolves as,

Circuit III :
$$|\psi_{\mathrm{cat}}(\pi/2, l\pi + (2N - l - r - Q)\pi/2)\rangle \longrightarrow |P, 0\rangle + (-1)^l |0, P\rangle ,$$

yielding the desired $N00N$ state, with $P = 2N - l - r - Q$. It may be noted that the superposition phase for the $N00N$-state at the output depends on the measurement record at the photodetectors. When $l < r$, the additional phase shift in Circuit II causes this phase to be $r\pi$ rather than $l\pi$.

Next, we estimate the fidelities of the states produced by our $N00N$ state generator, and clarify its behaviour for large photon number. To do this, we first compute the fidelity for one component of a cat state generated by Circuit I, which we denote by $|\psi_G(\Delta_0)\rangle$. We assume, as in Eq. (1), that the function $G(X)$, describing the localisation of the relative phase, assumes its Gaussian asymptotic form. Note that the rate of localisation is faster when phase correlations evolve at more than one value. We assume that the state at the input is the dual Fock state $|N\rangle_1 |N\rangle_2$, and that a total of $D = l + r$ detections have occurred. Then,

$$\begin{aligned}
F &\sim \left| \langle \psi_{\mathrm{G}}(\Delta_0) | \psi_\infty(\Delta_0) \rangle \right|^2 \\
&\simeq \frac{\left[\int_{-\pi/2}^{\pi/2} d\Delta \cos^S\left(\frac{\Delta}{2}\right) \exp\left(-\frac{D\Delta^2}{4}\right) \right]^2}{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} d\Delta d\Delta' \cos^S\left(\frac{\Delta - \Delta'}{2}\right) \exp\left[-\frac{D}{4}\left(\Delta'^2 + \Delta^2\right)\right]} \\
&\to \sqrt{1 - \frac{1}{\left[2\left(\frac{2N}{S}\right) - 1\right]^2}} ,
\end{aligned}$$

where $S = 2N - D$ is the total photon number. This result was derived assuming that S and D are not small. The value for the fidelity depends only on the ratio of detections to input photons. For example, when $D/2N$ is one half, $F = 0.94$, and when $D/2N$ is two thirds, $F = 0.98$. To verify this result, we computed numerically exact values for the fidelity, $\left| \langle \psi_{\mathrm{l,r}} | \psi_{\mathrm{cat}}(\Delta_0(l, r), l\pi) \rangle \right|^2$, for a range of states $|\psi_{l,r}\rangle$ generated by Circuit I. For input state $|3\rangle_1 |3\rangle_2$, the fidelity is 0.94 for $(l, r) = (1, 2)$ and $(2, 1)$, and anomalously it is 1 for $(l, r) = (1, 1)$. For input state $|5\rangle_1 |5\rangle_2$ the values are 0.94 and 0.96 when $l + r = 5$, and range from 0.96 to 1 when $l + r = 6$, while for input state $|15\rangle_1 |15\rangle_2$ the values range from 0.92 to 0.96 when $l + r = 15$, and from 0.96 to 0.99 when $l + r = 20$.

Finally, we performed a complete numerical simulation of the $N00N$-state generator, to verify that Circuits I, II and III work together as predicted. In particular, it was necessary to check that Circuits II and Circuit III function as expected when the relative phase correlations for the cat states are not perfectly well-defined. The results are shown in Fig. 3. Each point in the plots corresponds to a particular choice of input state, and measurement by Circuit I. The height corresponds to the expected photon number for the output $N00N$ state, and the color to its fidelity. Averages are taken over all possible outcomes to the third photodetection performed by Circuit II. For comparison, the mesh shows the predictions of the preceding analysis. Good agreement is seen between these analytical predictions and the numerical results. However, inspection of individual outcomes in Circuit II reveals that the high fidelities are not maintained in every case. Roughly speaking, improbable outcomes were often found to have low fidelity.
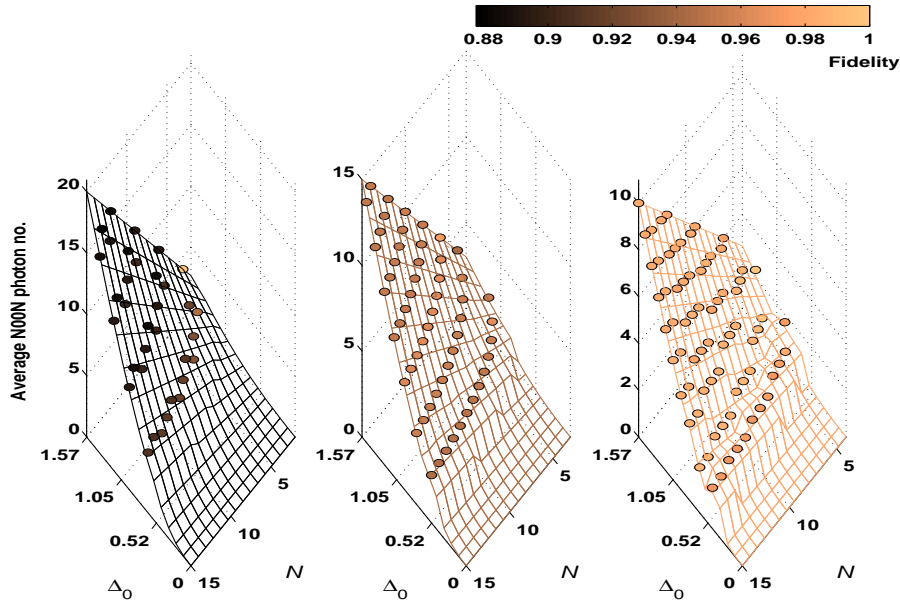
FIG. 3: Fidelities (color) and photon number (vertical axis) are displayed for outputs of our $N00N$ states generator. Each point corresponds to a possible outcome to Circuit I, for which D photons are detected. Input states $|N\rangle_1|N\rangle_2$ are considered for N up to 15. Going from left to right, $D/2N$ is one third, one half and two thirds.